ECON 897 Test (Week 4)
Aug 7, 2015

Important: This is a closed-book test. No books or lecture notes are permitted. You have **120** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

If you use want to use a theorem that we proved in class, be sure to say exactly which theorem you are using, state all of its assumptions and be sure that they are satisfied. Otherwise, you will be given partial credit.

1. **(15 points)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by:

$$g(x, y, z) = xy + yz + zx$$

$$f(x, y) = (xy, x\cos y, x\sin y)$$

   (a) **(5 points)** Are the functions f and g differentiable? Be sure to say why they are or why they aren't.

   Proof. Yes, they are because their partial derivatives exist and are continuous. □

   (b) **(5 points)** Define $h = g \circ f$. Find the representation matrix of $(Dh)_{(x,y)}$.

   Proof. Apply the chain rule. □

   (c) **(5 points)** Do the representation matrices of $(D^2 f)_{(x,y)}$ and $(D^2 g)_{(x,y,z)}$ exist? If they do, find them.

   Proof. The representation matrix for the second derivative of g exists, since the function goes to $\mathbb{R}$. However, there does not exist a matrix representation of $(D^2 f)_{(x,y)}$, since the function goes to $\mathbb{R}^3$. □

2. **(20 points)** Suppose there are n goods. To each price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^n_{++}$ corresponds a unique demand vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_{++}$ implicitly defined by the following n equations:

$$\begin{array}{rcl} U_1(x_1, \cdots, x_n) & = & p_1 \\ \vdots & \vdots & \vdots \\ U_n(x_1, \cdots, x_n) & = & p_n \end{array}$$

where $U = (U_1, \cdots, U_n) : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is continuously differentiable and the representation matrix of DU is a negative definite matrix.

(a) **(10 points)** Prove that if the price of ith good increases, then the demand for this good decreases, i.e. $\frac{\partial x_i}{\partial p_i} < 0$, $1 \leq i \leq n$. (Recall a symmetric matrix A is negative definite if for all $x \neq 0$, $x^T A x < 0$.)

Proof. Same as in homework. Here, I wanted you to explicitly say that, since the inverse of DU is negative definite, then for all $i \in \{1, \ldots, n\}$:

$$\frac{\partial x_i}{\partial p_i} = e_i^t (DU)^{-1} e_i < 0$$

□

(b) **(10 points)** If $n = 2$, find expressions for $\frac{\partial x_i}{\partial p_j}$, for $i, j \in \{1, 2\}$.

Proof. You just need to use the implicit function theorem and know how to find the inverse of a 2×2 matrix. □

3. **(15 points)** One of the separating hyperplane theorems that we proved was: Let $D \subseteq \mathbb{R}^n$ be <u>compact</u> and convex, and $E \subseteq \mathbb{R}^n$ be closed and convex. Assume $D \cap E = \emptyset$. Then, there exists a hyperplane $H(p, a)$ such that $p \cdot e < a$ for all $e \in E$ and $p \cdot d > a$ for all $d \in D$. Give an example of a case in which the set D is not compact and, therefore, there does not exist such a hyperplane.

Proof. I made a mistake on this question. I wanted to ask for an example in which the theorem fails if the word <u>compact</u> were replaced by the word <u>closed</u>. Think about this example!

As the question just asked to omit the word <u>compact</u>, the example is pretty easy. Take, for example, $[0, 1)$ and $[1, 2]$. These sets satisfy all the conditions for the theorem to apply, but cannot be strictly separated.

□

4. **(20 points)** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be homogeneous of degree 1 and quasi-concave. Prove that f is concave.

Remember that f is homogeneous of degree 1 if $f(\lambda x) = \lambda f(x)$, for all $\lambda \neq 0$.

Proof. Let $x, y \in \mathbb{R}^n$, and $\lambda \in (0,1)$. Since $f(x) > 0$ and $f(y) > 0$, there exists $\theta > 0$ such that $\theta f(\lambda x) = f((1-\lambda)y)$.

Since f is homogeneous of degree 1, this is equivalent to $f(\theta\lambda x) = f((1-\lambda)y)$. Consider the following linear combination between $\theta\lambda x$ and $(1-\lambda)y$:

$$\left(\frac{1}{1+\theta}\right)(\theta\lambda x) + \left(\frac{\theta}{1+\theta}\right)((1-\lambda)y)$$

Since f is quasi-concave:

$$\begin{aligned}
f\left(\left(\frac{1}{1+\theta}\right)(\theta\lambda x) + \left(\frac{\theta}{1+\theta}\right)((1-\lambda)y)\right) &= f\left(\left(\frac{\theta}{1+\theta}\right)(\lambda x) + \left(\frac{\theta}{1+\theta}\right)((1-\lambda)y)\right) \\
= \left(\frac{\theta}{1+\theta}\right) f\left((\lambda x) + ((1-\lambda)y)\right) &\geq \left(\frac{1}{1+\theta}\right) f(\theta\lambda x) + \left(\frac{\theta}{1+\theta}\right) f((1-\lambda)y) \\
&= \left(\frac{\theta}{1+\theta}\right) f(\lambda x) + \left(\frac{\theta}{1+\theta}\right) f((1-\lambda)y) \\
&\Leftrightarrow f\left((\lambda x) + ((1-\lambda)y)\right) \geq f(\lambda x) + f((1-\lambda)y)
\end{aligned}$$

$\square$

5. **(30 points)** Let $U : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ be continuous, quasi-concave and increasing[1]. Let $x^* \in \mathbb{R}^n_{++}$.

   (a) **(10 points)** Prove that there exists a $p \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $p \cdot x^* \leq M$ and $p \cdot x \geq M$ for all x such that $U(x) \geq U(x^*)$.

   Proof. U is quasi-concave if, and only if, the upper contour set, C_α, is convex for all $\alpha \in \mathbb{R}$, where C_α is defined as:

   $$C_\alpha = \{x \in \mathbb{R}^n | U(x) \geq \alpha\}$$

   Consider the upper contour set at $\alpha = U(x^*)$. Then, the set $C_{U(x^*)} = \{x \in \mathbb{R}_n | U(x) \geq U(x^*)\}$ is convex. Clearly, $x^* \notin int(C_{U(x^*)})$. Otherwise, there would exist an ϵ-ball around x^* such that $B(x^*, \epsilon) \subseteq C_{U(x^*)}$, which means that, for $\lambda < 1$ sufficiently close to 1, by continuity of U, $U(\lambda x^*) \geq U(x^*)$. This would contradict U being increasing.
   Thus, by the supporting hyperplane theorem, there exists $p \in \mathbb{R}^n$ and M, such that $p \cdot x^* \leq M$ and $p \cdot x \geq M$ for all x such that $U(x) \geq U(x^*)$. $\square$

[1] Definition: A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is increasing if $F(x) > F(y)$ whenever $x >> y$, where $x >> y$ means that $x_i > y_i$ for all $i \in \{1, \ldots, n\}$.

(b) **(10 points)** Prove that $p \cdot x > M$ if x is such that $U(x) > U(x^*)$.

Proof. Assume not. Then, there exists an x with $U(x) > U(x^*)$ and $p \cdot x = M$. Notice that by continuity of U, for $\lambda < 1$ sufficiently close to 1, $U(\lambda x) > U(x^*)$. Moreover, $p \cdot \lambda x = \lambda(p \cdot x) < M$, which cannot happen. □

(c) **(10 points)** Prove that, in fact, $p \in \mathbb{R}^n_+$.

Proof. Assume that there exists $i_0 \in \{1, \ldots, n\}$, such that $p_{i_0} < 0$. Fix $\lambda > 1$ and $x' = \lambda x^*$. Since U is increasing, $U(x') > U(x^*)$. Moreover, $U(x' + Ne_{i_0}) > U(x^*)$ for any $N > 0$, where e_{i_0} is the unit vector in the i_0-th component. Note that for N sufficiently large, $p \cdot (x' + Ne_{i_0}) < M$, which cannot happen. Thus, $p_i > 0$ for all $i \in \{1, \ldots, n\}$.

□